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A DEFICIENCY IN CURRENT FINITE ELEMENTS FOR THIN SHELL APPLICATIONS†

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Abstract—The paper investigates the accuracy of displacement finite elements developed for thin shell applications. A careful choice of shell theory is made and variational equations are used to obtain element equations. The results show that currently available finite elements use incomplete approximate forms for their displacement fields. The elements have been constructed so that either rigid body or "sensitive" solution modes are adequately represented but not both.

1. INTRODUCTION

THE finite element method is a piecewise application of the Rayleigh–Ritz technique and has been employed with great success in the numerical solution of a wide range of problems in continuum mechanics. At the present time this method is being extended to find solutions for loaded shells of arbitrary shape, and a large number of papers have already appeared on this subject [1]. Despite all this work no really systematic attempt has been made to examine the nature of the problem and kind of errors which may occur in dealing with shells in a piecewise manner.

The purpose of the present paper is to explore the possibility of obtaining accurate numerical solutions to general shell problems by using a finite element displacement analysis. In performing such an analysis the first critical decision which needs to be made lies in the selection of a set of constitutive equations for the underlying shell theory. According to Truesdell and Toupin [2] in a general field theory the set of constitutive equations must satisfy certain mathematical principles. With particular reference to shell theory Gol'denveizer [3] and Naghdi [4] have noted that from amongst these general principles there are three which have a particular significance. These three are:

(a) Consistency. Any set of constitutive equations should be consistent with the principles of energy and equilibrium. As Gol'denveizer points out this implies the existence of a reciprocity theorem analogous to Bettis' principle.

(b) Rigid displacement invariance. The equations should remain invariant under rigid body displacements. This does not mean that the strain measures necessarily produce zero strain for this kind of motion, but that the constitutive equations give rise to zero strain energy.

(c) Coordinate invariance. The equations should be stated by a rule which holds equally well in all coordinate systems. This condition can easily be satisfied if the appropriate equations are stated in tensorial form or by the aid of direct notations not employing coordinates at all.

Although these conditions are easily stated they impose severe limitations on any proposed new shell theory and are not necessarily easy to satisfy. In view of this, when finite element applications are considered, the best course would seem to lie in choosing a theory which already satisfies these conditions rather than construct a new one. This is the policy which is adopted in the present investigation where the modified Love–Kirchhoff theory proposed by Naghdi [5] is employed. It may be noted here that many finite shell elements used in the past relied on the theories of Love or Novozhilov [6], neither of which gives a satisfactory compliance with all the principles (a–c).

It may be argued that the degree of rigour created by the requirement that all the principles (a–c) are applied is unnecessary in view of the fact that numerical methods usually give approximate answers. Thus Zienkiewicz [1] and his co-workers, for example, have developed a shell element by using the isoparametric element without the imposition that the shell theory should conform to the Love–Kirchhoff hypothesis. In order to recover an approximate thin shell theory when the thickness is small certain *ad hoc* variations in the integration order are made. It is extremely difficult to relate these modifications to the requirements (a–c) and to the Koiter "consistency" requirements [7]. In view of this it would be difficult to maintain confidence in this type of element, for thin shell applications, in the absence of adequate confirmatory closed form or experimental results. However, in the present context, the principles are rigorously imposed in order to be sure that any errors which do arise come from the numerical approximation technique and not from an unsuitable shell theory.

In considering the adequacy or otherwise of displacement shell elements there are three important criteria to be considered. First of all the solution technique should be capable of dealing with the kind of "sensitive" solutions listed by Morley [7] some of which have caused problems in the past development of shell theory, an example of this class of problem is that of a slit cylinder under torsional loading. In this same context, the approximate displacement field must define a satisfactory description of the rigid-body displacement modes. Secondly, an element is required which fully satisfies the conditions of inter-element compatibility. For certain problems a fourth criterion requiring that the element must be suitable for deep shell applications would need to be imposed. Thus for general applications we have four conditions which must be satisfied by the finite element itself in addition to the three (a–c) imposed on the underlying shell theory.

One of the earliest and simplest displacement finite element systems to be developed for use in solving shell problems considered the shell as an assembly of flat elements [1, 8]. Each element has an approximate displacement field which can give rise to both membrane and couple stress resultants and the element is a natural extension of the studies done on plate bending elements. Apart from the advantage of simplicity the flat element can handle rigid-body displacements without incurring any errors and can be made fully compatible. However, in practice there is always a coupling action between the bending and membrane stress fields and for certain problems this is unacceptable. If, for example, the element was used to analyse the torsion of a slit cylinder it is difficult to see how it could achieve the correct answer which requires that all the stresses be negligible everywhere with the exception of the twisting moment. This apparent inability to handle certain shell problems imposes a limitation on the general application of flat elements since it would require knowing in advance whether any particular solutions contained dominant components from these missing solutions.

Another simple approach is to consider the shell as an assembly of elements each of which is a portion of a shallow shell [9]. Once again the elements can be made fully conforming in the displacement field and in principle can give a satisfactory description to the rigid-body modes but not to "sensitive" solutions. In addition there are two further objections. First of all adjacent elements which are portions of different parabolic surfaces will not match properly and will produce discontinuities in the shell surface. Secondly if a sequence of shallow shell elements are used to approximate a deep shell then the net result is equivalent to using shallow shell theory for the solution of deep shell problems.

A further possibility is to employ curved finite elements with an appropriate general shell theory and to write the equations in terms of a flat two-dimensional Euclidean reference surface and a variable representing the perpendicular distance of the curved surface from the flat one [10]. This kind of transformation is easily performed and the appropriate tensorial equations may be found in Green and Zerna [11]. By using a reference plane it is possible to gain all the advantages of a flat element system but without sacrificing the true shape. Thus the compatibility and rigid-body criteria are satisfied as they are with the flat element but, unlike the flat element, a preliminary inspection reveals no artificial coupling between the couple and membrane stress resultants. However, because we are mapping between a curved and a flat surface the process which changes the transcendental rigid-body displacements into polynomial functions also transforms the polynomial "sensitive" solutions into transcendental form. Thus, as we shall see later, the same situation exists for this element as with an element embedded in the shell surface, but there are other difficulties in the present case. Let us consider the case where the corner nodes for the elements lie on the curved surface and the reference surface for each element is formed by passing a plane through each node (the nodes must be co-planar). If the shell being analysed has a non-zero Gaussian curvature, Fig. 1(a), it is immediately clear that projecting normally

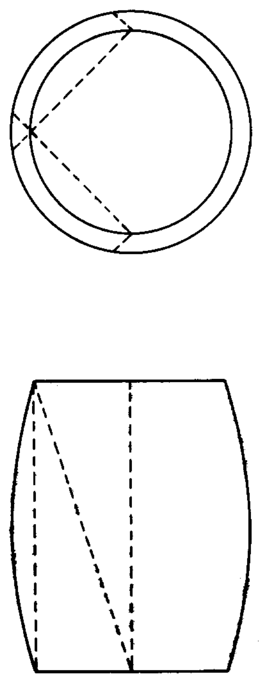

FIG. 1(a). Discontinuities caused by element reference surface.

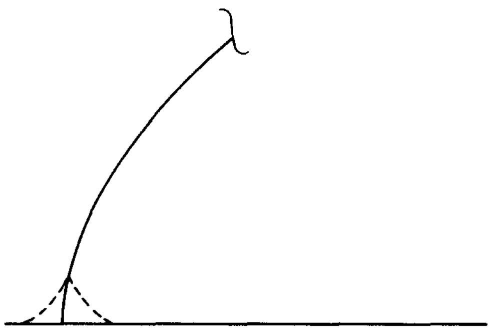

FIG. 1(b). Cusp at junction of reference and shell surfaces.

from each reference surface does not cover the shell surface. It is assumed by Visser [12] that the existence of this kind of void does not give rise to any real difficulties. However, if a large problem is being solved and certain regions need to be covered by a few elements such voids could well be large. Further it is not clear what kind of effect these voids will have on "sensitive" solutions since the kind of coupling experienced by flat elements will also exist in this particular situation. If it is decided to circumvent this difficulty by having just one reference plane, Fig. 1b, problems will still arise if the shell and the reference surface intersect.

An alternative element which falls partially into the above category is that of Argyris and Scharpf [13]. However, this element has been discussed by Dupuis [14] to which reference should be made for an assessment.

In view of all these doubts it would seem that the only suitable displacement element for deep shells is one embedded in the curved shell surface and described in terms of surface coordinates. Such an element can be easily constructed to satisfy inter-element conformity but then runs into rigid-body displacement errors. The most usual procedure with such an element is to describe the tangential and normal displacement fields by polynomial functions of the coordinates. Unfortunately for curved surfaces rigid-body movements are properly expressed as transcendental functions of the surface coordinates as opposed to polynomials. It has been suggested by Cantin and Clough [15] that a mixed polynomial–transcendental form should be taken as the approximate displacement field in order that rigid-body modes be explicitly represented, but this removes some of the "sensitive" solution modes and would give rise to errors if these components appeared in any problem. If we accept that polynomial approximation is the only satisfactory answer we may turn to Morley's paper [7] where it is noted that if the tangential displacements are limited to quadratic expressions then, in order to retain the same kind of accuracy associated with first approximation shell theory, it is generally necessary that the linear dimensions of the finite element be of the same order of magnitude as the shell thickness in addition to a requirement that the rigid-body movement is accommodatingly small. It is further noted [7] that with quartic expressions for these displacement components the minimum linear dimension increases to $\sqrt{(R \times \text{the thickness})}$, where R is the current minimum radius of curvature. Thus simply increasing the order of the approximating polynomials does not remove rigid-body errors but reduces their influence. These remarks of Morley play a central rôle in the present paper where two elements are presented, each employing an approximating polynomial for the components of the displacement field, and each is used to solve "sensitive" problems. Since the work forms an investigation into accuracy, the elements are rectangular but they are constructed so that their edges need not lie along lines of principle curvatures and they can treat shells with continuously varying radii.

2. THE ELEMENTS

The elements developed for the present investigation are constructed so that they are capable of analysing a doubly curved shell with continuously varying radii. However, the analytic "sensitive" solutions available for comparison with finite element results are obtained for circular cylindrical shells. A considerable simplification in the algebra is achieved if the equations for the finite elements are written in terms of a circular cylindrical configuration as opposed to a general shape. This simplification is adopted below where the general equations actually employed by the element are not presented but only the form that they take when specialized to the circular cylindrical case. We may note, in passing, that circular cylindrical shells exhibit all the properties of a more general shape and the results obtained below are equally valid for all types of surfaces.

In the theory of Naghdi [5] the symmetric components of median strain and curvature change for a circular cylindrical shell are expressed in terms of the displacement components by

$$\begin{aligned}
\gamma_{(\theta\theta)} &= \frac{1}{R}\frac{\partial v}{\partial \theta}+\frac{w}{R}, \\
\gamma_{(xx)} &= \frac{\partial u}{\partial x}, \\
\gamma_{(x\theta)} &= \gamma_{(\theta x)} = \frac{1}{2}\left\{\frac{\partial v}{\partial x}+\frac{1}{R}\frac{\partial u}{\partial \theta}\right\}, \\
\hat{\varkappa}_{\theta\theta} &= -\frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2}+\frac{2}{R^2}\frac{\partial v}{\partial \theta}+\frac{w}{R^2} \\
\hat{\varkappa}_{xx} &= -\frac{\partial^2 w}{\partial x^2} \\
\hat{\varkappa}_{x\theta} &= \hat{\varkappa}_{\theta x} = -\frac{1}{R}\frac{\partial^2 w}{\partial x \partial \theta}+\frac{1}{R}\frac{\partial v}{\partial x}.
\end{aligned} \tag{2.1}$$

The shell geometry and sign conventions for displacements, loads, stress resultants and stress couples are shown in Fig. 2. The symmetric pseudo-stress resultants and stress couples which correspond to (2.1) are given by:

$$\begin{aligned}
\hat{N}_{\theta\theta} &= N_{\theta\theta}-\frac{M_{\theta\theta}}{R}, \\
\hat{N}_{xx} &= N_{xx}, \\
\hat{N}_{x\theta} &= \hat{N}_{\theta x} = N_{x\theta}-\frac{M_{x\theta}}{R} = N_{\theta x}, \\
M_{(\theta\theta)} &= M_{\theta\theta}, \\
M_{(xx)} &= M_{xx}, \\
M_{(x\theta)} &= \tfrac{1}{2}\{M_{x\theta}+M_{\theta x}\},
\end{aligned} \tag{2.2}$$

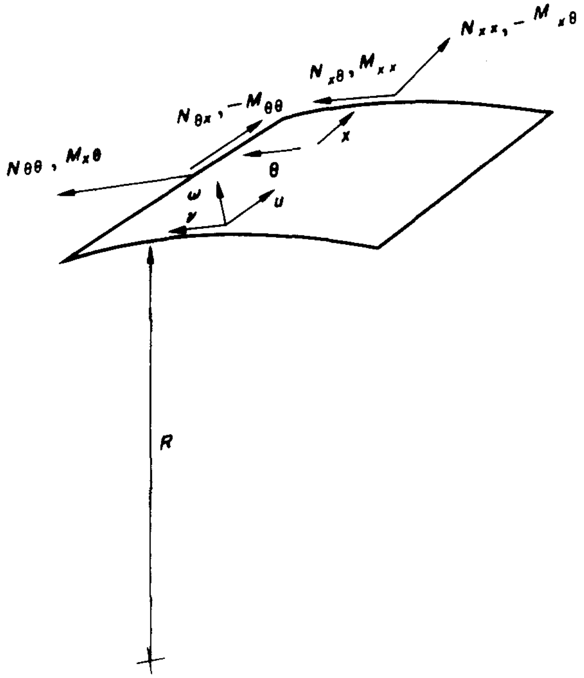

FIG. 2. Notation for shell theory.

and the constitutive equations become,

$$
\begin{aligned}
\hat{N}_{\theta\theta} &= \frac{Eh}{(1-v^2)}(\gamma_{(\theta\theta)} + v\gamma_{(xx)}) \\
\hat{N}_{xx} &= \frac{Eh}{(1-v2)}(\gamma_{(xx)} + v\gamma_{(\theta\theta)}) \\
\hat{N}_{x\theta} = \hat{N}_{\theta x} &= \frac{Eh}{1+v}\gamma_{(x\theta)} \\
M_{(\theta\theta)} &= \frac{Eh^3}{12(1-v2)}(\hat{\varkappa}_{\theta\theta} + v\hat{\varkappa}_{xx}) \\
M_{(xx)} &= \frac{Eh^3}{12(1-v2)}(\hat{\varkappa}_{xx} + v\hat{\varkappa}_{\theta\theta}) \\
M_{(x\theta)} = M_{(\theta x)} &= \frac{Eh^3}{12(1+v)}\hat{\varkappa}_{x\theta}
\end{aligned}
\tag{2.3}
$$

where h is the shell thickness, E Young's modulus and v Poisson's ratio.

Once a particular shell theory has been selected the application of the finite element method becomes one of obtaining an appropriate set of linear simultaneous equations. The selection of these equations can be made in an elegant and rigorous manner by employing the well tried variational principles of elasticity theory [16–18]. Thus the potential energy theorem may be applied to a single discrete element in a collection of connected finite elements on the assumption that we are dealing with a displacement element which fully satisfies interelement compatibility and that the strain field corresponds to the displacement field. If it is specialized to the case of a cylindrical shell with the Naghdi symmetric terms the theorem states that the variation of a functional J is identically zero where

$$J = \int_A \left[-\frac{h}{2}\left\{\{\gamma\}^T\{A\}\{\gamma\} + \frac{h^2}{12}\{\hat{\varkappa}\}^T\{A\}\{\hat{\varkappa}\}\right\} + p_\theta v + p_x u + pw \right] \mathrm{d}A$$

$$+ \int_c (N_n^* u_n + N_{ns}^* u_s + M_n^* \phi_n + V_n^* w)\,\mathrm{d}s. \tag{2.4}$$

In equation (2.4) the column vectors $\{\gamma\}$ and $\{\hat{\varkappa}\}$ consist of the strain and curvature change measures given in (2.1), the remaining terms are given by

$$\{A\} = \frac{E}{(1-\nu^2)}\begin{pmatrix} 1 & 0 & \nu \\ 0 & (1-\nu) & 0 \\ \nu & 0 & 1 \end{pmatrix}$$

$$u_n = u\cos\alpha + v\sin\alpha$$

$$u_s = -u\sin\alpha + v\cos\alpha$$

$$\phi_n = -\frac{\partial w}{\partial n} + (u_s\cos\alpha + u_n\sin\alpha)\sin\alpha$$

where α is the angle between the normal and the generator and with p_θ, p_x and p external applied surface loads. Any applied concentrated loads in equation (2.4) are assumed to be included in the shear stress resultant V_n^*. The set of approximate equations for the entire shell are then obtained by taking the variation of J with respect to the displacements, v, u and w for all the elements making up the shell. Before leaving (2.4) it may be noted that by keeping the non-symmetric resultants and couples in the contour integral we are able to apply external edge loads to the finite element system in a straightforward manner.

Let us assume that a shell surface is divided up into a series of rectangular domains by means of a set of orthogonal curvilinear coordinates and that each of these domains constitutes a displacement finite element for which we require a stiffness matrix corresponding to the modified Naghdi theory given above. It is further assumed that on each of these domains an approximate displacement field can be generated by using only the displacements, and their derivatives, at each corner of the rectangle together with the appropriate Hermite polynomial.

Within the finite element discipline the use of the Hermite polynomials for generating approximate fields is not new and an adequate discussion for this type of application can be found in an expository paper by Bogner, Fox and Schmit [19]. In the specific case where the finite element technique has been used to analyse shell problems this method of approximation has been exploited by Key and Beisinger [20].

The first rigorous definition of a bivariate generalization of the Hermite interpolation formula seems to have been given by Alhlin [21] who shows that a suitable approximate form for a function $f(x, y)$ is given by

$$f_m(x, y) = \sum_{i=1}^{n} \sum_{r=1}^{n} h_r(x) g_i(y) f(x_r, y_i) + \sum_{i=1}^{n} \sum_{r=1}^{n} h_r(x) \bar{g}_i(y) \frac{\partial}{\partial y} f(x_r, y_i)$$
$$+ \sum_{i=1}^{n} \sum_{r=1}^{n} \bar{h}_r(x) g_i(y) \frac{\partial}{\partial x} f(x_r, y_i) + \sum_{i=1}^{n} \sum_{r=1}^{n} \bar{h}_r(x) \bar{g}_i(y) \frac{\partial^2}{\partial x \partial y} f(x_r, y_i) \qquad (2.5)$$

where

$$h_r(x) = [1 - 2l_r'(x)(x - x_r)][l_r(x)]^2$$

$$\bar{h}_r(x) = (x - x_r)[l_r(x)]^2$$

$$g_i(y) = [1 - 2m_i'(y)(y - y_i)][m_i(y)]^2$$

$$\bar{g}_i(y) = (y - y_i)[m_i(y)]^2$$

and l_r and m_i are the Lagrangian coefficients,

$$l_r(x) = \lambda(x)/(x - x_r)\lambda'(x_r)$$

$$m_i(x) = \mu(y)/(y - y_i)\mu'(y_i)$$

with

$$\lambda(x) = \prod_{i=1}^{n} (x - x_i)$$

$$\mu(y) = \prod_{i=1}^{n} (y - y_i).$$

The expression (2.5) defines the approximate form $f_m(x, y)$ in terms of known values of the function $f(x, y)$ and certain of its derivatives at a given set of mesh points $x_r, y_i (i = 1 \ldots n, r = 1 \ldots n)$. It should be noticed that this type of approximation *exactly* represents functions of the form

$$f_m(x, y) = \sum_{i=1}^{k} a_k x^p y^p \qquad p \leq 3$$

where k is any positive integer and the a_k's are a set of constants.

Returning to the finite element, there are three components of the displacement field to be considered, the "in-plane" displacements v, u and the displacement normal to the shell surface w. In the present analysis two kinds of elements are employed and the approximation chosen for one is that v, u should be first order interpolation polynomials and w second order giving a 24×24 stiffness matrix for each element. The second element which has a 48×48 stiffness matrix uses the same second order interpolation polynomial for all the displacement components.

Considering the element as a domain with orthogonal sides we may prescribe values of the variables at the corners (nodes) of the region and construct approximating Hermitian interpolation functions. These approximate forms are then substituted into the functional

(2.4) and the appropriate matrix equations are obtained by taking the variation of J for each of the prescribed values in turn.

3. PRESENTATION AND DISCUSSION OF RESULTS

The equations used for the present analysis are such that the elements can be used for shells with continuously varying radii. If they are specialized to the case of a cylindrical shell with the element edges coincidental with the principle curvatures it is clear that the (48×48) stiffness matrix is simply that used by Bogner, Fox and Schmitt [22]. The only difference being that the present element uses numerical integration whilst Bogner and his associates employ an analytic form. Using the examples in [22] the results obtained by the present (48×48) element using sixteen integration points are identical with those of Bogner, Fox and Schmitt. This identity indicates that the potential energy of the two systems is the same and does not imply that the constitutive equations actually used for stress calculations are the same. In fact, Bogner *et al.* do not calculate any stress resultants or couples and do not present a set of constitutive equations. In the course of the following sections a comparison of certain stress components is made which necessitates the use of an adequate set of constitutive equations. It is precisely this requirement, in conjunction with the mathematical principles laid down in the introduction, which led to the selection of the Naghdi theory.

In the case of the (24×24) element there is no equivalent element available for direct comparison and in this case element integrity is demonstrated with the aid of two small examples. These are shown in Fig. 3 and consist of a square plate with encastré edges under

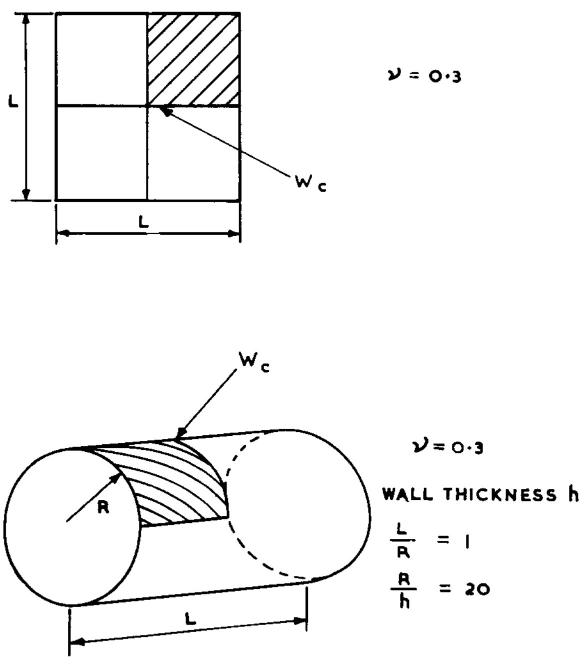

FIG. 3. Plate and shell tests for element integrity.

a central concentrated load and a circular cylindrical shell with encastré ends under uniform pressure. The accuracies achieved by the element are shown in Table 1 where the column giving the number of elements indicates how many elements were used to cover the shaded areas in Fig. 3.

TABLE 1. PLATE AND CYLINDER TESTS FOR 24 DEGREES OF FREEDOM ELEMENT

Problem	Number of elements	Number of displacement unknowns	% error in w_c compared with classical value
Plate	1	16	−7·1
Plate	4	36	−3·2
Cylinder	1	24	+2·0
Cylinder	4	60	−0·5

Having illustrated that both elements work in a reasonably satisfactory manner attention is now fixed on the "sensitive" examples discussed in the introduction. One such problem, that of a slit circular cylinder under torsional loading, has played an important rôle in highlighting the inherent defects in shell theories. In the present context it will be used to show that the arguments deployed by Morley [7], concerning the restrictions on element size imposed by the approximating polynomial, are borne out in practice. The problem has a very simple solution, the only significant contribution to the stress field being a constant torsional couple and the displacement field has no component normal to the surface but the other two components have a linear variation in the surface co-ordinates. The method of attack is to solve the problem twice with each element, first with the rigid-body modes secured so that no displacements normal to the shell are possible, and secondly with the rigid-body modes secured so that normal displacements are permitted. The problem is completely specified by traction boundary conditions and the equilibrium set requires not only tangential edge forces but also loads normal to the surface as shown in Fig. 5. Since both elements can accurately approximate a linear displacement field it is anticipated that the problem with the first set of rigid-body conditions will be accurately solved no matter how large the element becomes. In the second case with the equilibrium set the normal displacement should give rise to no strain and is then equivalent to a rigid-body motion. With this situation, according to Morley, inaccuracies should appear with the present elements when the element dimensions exceed the shell thickness for the (24×24) case and when they exceed $\sqrt{(R \times \text{thickness})}$ for the (48×48) element. We note at this point that elements using the flat two-dimensional reference surface [10] will give rise to the above mentioned inaccuracies for both types of securing condition for the reasons given in the introduction.

The problem is shown in Fig. 4 and for the numerical examination a strip is taken from the circular cylindrical shell as indicated in the figure by the dotted lines. In the case of both the (24×24) and (48×48) element this strip is approximated by a single element. A typical single element is shown in Fig. 5 where the generalized loads F_1 and F_2 both balance the externally applied torque and maintain the element in equilibrium. With the application of the loads F_2 the rigid-body modes may be secured by a method which allows a normal displacement. The particular securing system used here is shown in Fig. 5. An alternative method which does not permit normal displacements is achieved by putting $w = 0$ at all

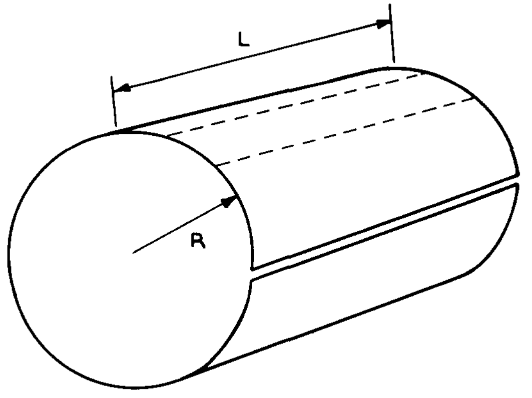

FIG. 4. Slit cylinder test.

four nodes and $u = v = 0$ at one node (see Fig. 5 for definition of the displacements w, u and v). The cylinder is assumed to have a thickness/radius (h/R) ratio of 0·01, the non-dimensional element length (L/R) is constant at 0·05 and various values are chosen for the angle ϕ which is subtended by the element. The torque parameter T/Gh^3 is constant at $1{\cdot}638 \times 10^{-3}$ where T is the actual applied torque and G is the shear modulus. Poisson's ratio for the material is taken as 0·3. With this loading condition the classical shell theory solution has only one significant stress field component which is $M_{x\phi}$ with a value of 500. In first approximation shell theory any stress resultant (N_{xx}, $N_{\phi\phi}$, $N_{x\phi}$) or couple (M_{xx}, $M_{\phi\phi}$)

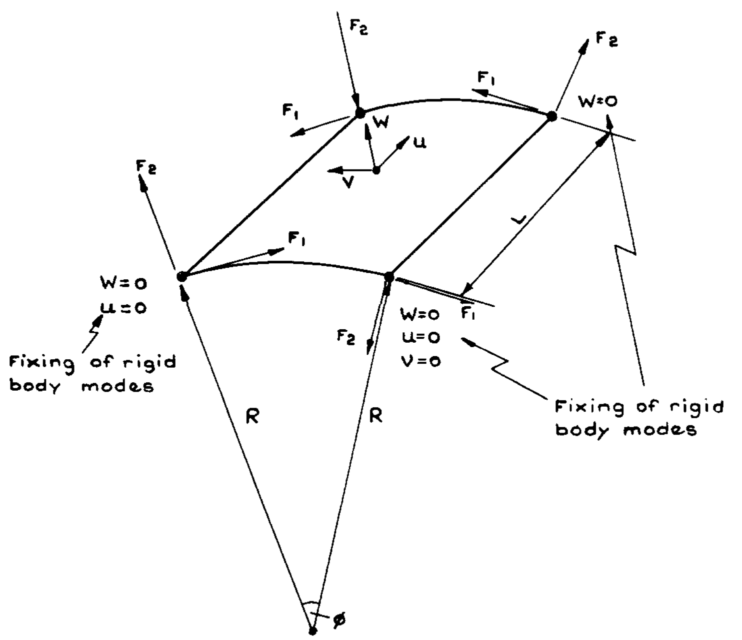

FIG. 5. Loaded strip for slit cylinder test.

TABLE 2(a). TORSION OF A SLIT CYLINDER. ELEMENT WITH 24 DEGREES OF FREEDOM, RIGID-BODY MODES SECURED IN THE MANNER SHOWN IN FIG. 4

ϕ rad.	$\theta \times 10^5$ for element	$\theta \times 10^5$ classical	$M_{x\phi}$ for element	Max[N_{xx}, $N_{\phi\phi}$, $N_{x\phi}$] for element	$M_{x\phi}$ classical	Max[N_{xx}, $N_{\phi\phi}$, $N_{x\phi}$] classical
0·01	7·77	7·80	496·43	$O[5]$	500·00	$O[5]$
0·1	7·16	7·80	460·00	$O[M_{x\phi}]$	500·00	$O[5]$
0·15	6·38	7·80	452·21	$O[M_{x\phi}]$	500·00	$O[5]$
0·20	5·57	7·80	448·37	$O[M_{x\phi}]$	500·00	$O[5]$

critical $\phi = h/R = 0{\cdot}01$

TABLE 2(b). TORSION OF A SLIT CYLINDER. ELEMENT WITH 24 DEGREES OF FREEDOM, RIGID-BODY MODES SECURED BY SETTING $w = 0$ AT ALL FOUR NODES AND $v = u = 0$ AT ONE

ϕ rad.	$\theta \times 10^5$ for element	$\theta \times 10^5$ classical	$M_{x\phi}$ for element	Max[N_{xx}, $N_{\phi\phi}$, $N_{x\phi}$] for element	$M_{x\phi}$ classical	Max[N_{xx}, $N_{\phi\phi}$, $N_{x\phi}$] classical
0·01	7·80	7·80	496·93	$O[0{\cdot}1]$	500·00	$O[5]$
0·1	7·80	7·80	496·93	$O[0.1]$	500·00	$O[5]$
1·571	7·80	7·80	496·93	$O[0.1]$	500·00	$O[5]$

critical $\phi = h/R = 0{\cdot}01$

is not considered as making a significant contribution to the stress field for this problem providing it has a value less than or equal to $5\,(M_{x\phi}h/R)$. Therefore, we may say that a finite element solution to the problem is correct if the value of $M_{x\phi}$ is near to 500 and no other couple or stress resultant has a value greater than 5.

The results obtained by securing the rigid-body modes in the manner shown in Fig. 4 are displayed in Table 2a. They clearly show how inaccuracies in the predicted twist per unit length θ increase as a function of the angle ϕ. The only error free result is the case where the element length in the ϕ direction is equal to the shell thickness and, when this length is increased to $10 \times$ thickness, large errors appear. It is at this stage that the advantages, indicated in the introduction, of using Naghdi's theory are seen. Because this theory has a set of adequate constitutive equations we can be sure that the erroneous couple stresses do not originate in the underlying shell theory but must be attributed to the use of an approximation in the solution techniques. In Table 2(b) the same element is used but the rigid-body modes are secured by the alternative configuration of setting the normal displacement $w = 0$ at all four nodes and $u = v = 0$ at only one node. In this case there

TABLE 3(a). TORSION OF A SLIT CYLINDER: ELEMENT WITH 48 DEGREES OF FREEDOM, RIGID-BODY MODES SECURED IN THE MANNER SHOWN IN FIG. 4

ϕ rad.	$\theta \times 10^5$ for element	$\theta \times 10^5$ classical	$M_{x\phi}$ for element	Max[N_{xx}, $N_{\phi\phi}$, $N_{x\phi}$] for element	$M_{x\phi}$ classical	Max[N_{xx}, $N_{\phi\phi}$, $N_{x\phi}$] classical
0·1	7·77	7·80	498·05	$O[0{\cdot}1]$	500·00	$O[5]$
0·2	7·74	7·80	496·34	$O[5]$	500·00	$O[5]$
0·3	7·60	7·80	493·25	$O[20]$	500·00	$O[5]$

critical $\phi = \sqrt{(Rh)} = 0{\cdot}1$

TABLE 3(b). TORSION OF A SLIT CYLINDER ELEMENT WITH 48 DEGREES OF FREEDOM, RIGID-BODY MODES SECURED BY SETTING $w = 0$ AT ALL FOUR NODES AND $v = u = 0$ AT ONE

ϕ rad.	$\theta \times 10^5$ for element	$\theta \times 10^5$ classical	$M_{x\phi}$ for element	$\text{Max}[N_{xx}, N_{\phi\phi}, N_{x\phi}]$ for element	$M_{x\phi}$ classical	$\text{Max}[N_{\phi\phi}, N_{xx}, N_{x\phi}]$ classical
0·01	7·80	7·80	499·13	$O[0{\cdot}1]$	500·00	$O[5]$
0·1	7.80	7·80	499·18	$O[0{\cdot}1]$	500·00	$O[5]$
1·571	7·80	7·80	499·18	$O[0{\cdot}1]$	500·00	$O[5]$

critical $\phi = \sqrt{(Rh)} = 0.1$

are no rigid-body type motions and the only displacement field which required defining is that appropriate to the analytic solution. The resulting linear displacement field is adequately represented by the approximate field and the element gives satisfactory results for *any* value of ϕ.

The same problem is re-analysed with the aid of the (48 × 48) and the results are presented in Tables 3(a) and 3(b). Once again the pattern is repeated in a presence of a trigonometric field (Table 3a) the results are satisfactory for values of ϕ up to 0·2 rad. i.e. 20 × thickness. In the presence of a linear displacement field [Table 3(b)] the results are also satisfactory for any value of ϕ.

The results obtained from the slit cylinder justify the arguments in the introduction, namely that displacement elements embedded in the curved surface employing a polynomial approximation scheme are accurate for "sensitive" problems but obey the "Morley rules" for rigid-body motions.

In order to test that elements using the flat two-dimensional reference surface also obey the same "rules", but in this case for errors arising in the sensitive solution modes, the same slit cylinder problem was solved using the triangular elements of Argyris and Scharpf [13]. This element uses a fifth order polynomial approximation to the displacement field and appears in two forms. Sheba 6 has 63 degrees of freedom, eighteen at each vertex and three at the midpoint of each side; Sheba 3 has 54 degrees of freedom with 18 at each vertex. As before a strip was taken (Figs. 4 and 5) from the cylinder but in this case the rectangular piece was approximated by joining two of the triangular elements. Although this element is free from rigid-body difficulties the results of the tests that are given in Table 4 were obtained with the rigid-body modes secured by putting $w = 0$ at the four corners of the rectangle and $u = v = 0$ at one corner. Once again the numerical results confirm the theoretical predictions in that significant stress resultants do not arise for sufficiently small element size but do occur for large elements. Three results are given for Sheba 3 and one for Sheba 6 and although the latter gives better results the same pattern occurs with both.

A second problem in the same class of "sensitive" solutions is that of the pure bending of a segment of a circular cylinder as shown in Fig. 6. In order to test this configuration the segment is assumed to be encastré at A (Fig. 6) with the loads applied at B. This system of supports gives rise to a trigonometric rigid-body type displacement field and it is to be expected that the correct solution, $M_{\phi\phi}$ a constant with no other significant stress contribution, will only be achieved for sufficiently small elements. The results given in Table 5 show that the "rules" are once more obeyed. With a suitable rigid-body fixing mode the correct solution of w and $M_{\phi\phi}$ both constant with all other stresses insignificant is achieved.

TABLE 4. TORSION OF A SLIT CYLINDER ANALYSED BY THE SHEBA ELEMENT

SHEBA 3

ϕ rad.	$\theta \times 10^5$ for element	$\theta \times 10^5$ classical	$M_{x\phi}$ for element	Max[$N_{\phi\phi}$, N_{xx}, $N_{x\phi}$] for element	M_x classical	Max[$N_{\phi\phi}$, N_{xx}, $N_{x\phi}$] classical
0·01	7·80	7·80	500·00	$O[0{\cdot}1]$	500·00	$O[5]$
0·1	7·80	7·80	500·00	$O[4]$	500·00	$O[5]$
1·571	3·574	7·80	366·00	$O[1000]$	500·00	$O[5]$

SHEBA 6

ϕ rad.	$\theta \times 10^5$ for element	$\theta \times 10^5$ classical	$M_{x\phi}$ for element	Max[$N_{\phi\phi}$, N_{xx}, $N_{x\phi}$] for element	M_x classical	Max[$N_{\phi\phi}$, N_{xx}, $N_{x\phi}$] classical
1·571		7·80	Range 497·89–502·38	$O[55]$	500·00	$O[5]$

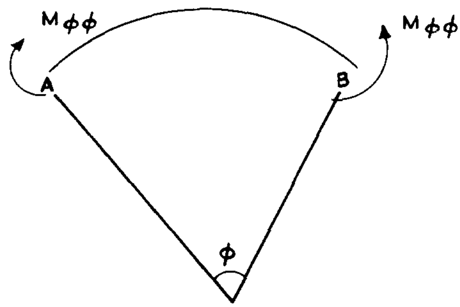

FIG. 6. Pure bending of a segment of a cylindrical shell.

TABLE 5. PURE BENDING OF A SEGMENT ANALYSED BY THE SHEBA ELEMENT. 24 DEGREES OF FREEDOM ELEMENT

ϕ rad.	$M_{\phi\phi}$ for element	Max[N_{xx}, $N_{\phi\phi}$, $N_{\phi x}$] for element	$M_{\phi\phi}$ classical	Max[N_{xx}, $N_{\phi\phi}$, $N_{\phi x}$] classical
0·01	498·54	$O[5]$	500·00	$O[5]$
0·1	509·08	$O[M_{\phi\phi}]$	500·00	$O[5]$

critical $\phi = h/R = 0{\cdot}01$

48 DEGREES OF FREEDOM ELEMENT

ϕ rad.	$M_{\phi\phi}$ for element	Max[N_{xx}, $N_{\phi\phi}$, $N_{\phi x}$] for element	$M_{\phi\phi}$ classical	Max[N_{xx}, $N_{\phi\phi}$, $N_{\phi x}$] classical
0·1	499·97	$O[1]$	500·00	$O[5]$
0·2	502·05	$O[5]$	500·00	$O[5]$
0·3	506·20	$O[9]$	500·00	$O[5]$

critical $\phi = \sqrt{(hR)} = 0{\cdot}1$

4. CONCLUSIONS

In the previous sections it is shown how current displacement thin shell elements are deficient in their displacement field representation. Either they can give an exact representation to the polynomial displacement fields occurring with "sensitive" solutions and are deficient in the trigonometric terms required for rigid-body representation. Or a trigonometric approximation scheme is employed that exactly represents the rigid-body fields but is deficient in "sensitive" solution components. The only exception to this being elements employing the flat reference plane where the argument must be inverted. In reviewing the elements currently available it can be seen that their proposers have made a choice and have elected to represent one or the other of these displacement fields but not both.

In view of the importance of "sensitive" solutions, and their relevance in demonstrating the completeness of a given approximation scheme for a displacement element, it would seem prudent for future proposers of shell elements to demonstrate their capacity to deal with these problems. If a conforming element was proposed which gave satisfactory results for problems like the slit cylinder case with both methods of securing the rigid-body modes it could then be used with confidence for any thin shell problem.

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Абстракт—В работе исследуется точность смещений конечных элементов, разработанных с целью применения к тонким оболочкам. Производится внимательный выбор теории оболочек. С целью получения уравнений элемента используются вариационные уравнения. Результаты указывают что общепринятые доступные конечные элементы применяют неполные приближенные выражения для полей их перемещений. Строится такие же элементы, чтобы было возможно уцовлетворительно представить или жесткое тело или типы "чувствительных" решений, но не оба.